Combinatorics in Banach space theory

PROBLEMS (Part 1)*

● **PROBLEM 1.1.** We say that a Banach space X is *complemented in its bidual* whenever $i(X)$ is a complemented subspace of X^{**}, where $i\colon X \to X^{**}$ stands for the canonical embedding.

(a) Show that every dual Banach space $X = Y^*$ is complemented in its bidual via a norm one projection.
(b) Show that a Banach space is complemented in its bidual whenever it is complemented in some dual Banach space.

Remark. Concerning (a): There are Banach spaces which are complemented in their biduals although not being isomorphic to any dual space, e.g. the space $L_1(0,1)$ (see [F. Albiac, N.J. Kalton, *Topics in Banach Space Theory*, Prop. 6.3.10]). The fact that it is not isomorphic to any dual space was proved by I.M. Gelfand in 1938 and involves the *Radon–Nikodým property* (see [J. Diestel, J.J. Uhl, *Vector Measures*, Cor. III.3.9]).

Concerning (b): This was probably first observed by J. Lindenstrauss in 1964.

● **PROBLEM 1.2.** Show that being complemented in its bidual is an isomorphic invariant, that is, if X and Y are Banach spaces, $X \simeq Y$ and X is complemented in X^{**}, then Y is complemented in Y^{**}.

Hint. Use adjoint operators.

● **PROBLEM 1.3.** Prove that a subset A of a Banach space X is relatively weakly compact if and only if it is bounded and the w^*-closure of $i(A)$ lives in X (here $i\colon X \to X^{**}$ is the canonical embedding). Use this fact to show that a set $A \subset X$ is weakly compact provided that for each $\varepsilon > 0$ there is a weakly compact set $A_\varepsilon \subset X$ with $A \subset A_\varepsilon + \varepsilon B_X$ (B_X is the unit ball of X).

Hint. Use the fact that the canonical embedding $i\colon X \to X^{**}$ yields a homeomorphism between the topological spaces $(X, \sigma(X, X^*))$ and $(i(X), \sigma(X^{**}, X^*))$, and combine it with the Banach–Alaoglu theorem. It is also useful here to know that the weak closure of a norm bounded set is norm bounded as well (why?).

Remark. The latter assertion of this problem is attributed to A. Grothendieck.

● **PROBLEM 1.4.** Let X be a separable Banach space. Show that every weakly compact subset of X^* is norm separable. This implies, in particular, that if X is separable and X^* is non-separable, then X^* is not WCG (e.g. $\ell_\infty = \ell_1^*$ is not WCG).

Hint. Use the first assertion of Problem 1.3 (without the word 'relatively') and the homeomorphism mentioned in the hint to that problem. Next, recall the classical fact saying that if X is a separable Banach space then every w^*-compact set $K \subset X^*$ is metrisable in the weak* topology.

Remark. The metric on K in the above mentioned, 'classical' fact may be given explicitly by

$$\rho(x^*, y^*) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x^* x_n - y^* x_n|,$$

*Evaluation: ●=2pt, ●=3pt, ●=4pt

where $\{x_n\}_{n=1}^{\infty}$ is any dense subset of the unit ball B_X. Note that (by the Banach–Alaoglu theorem) $K \subset X^*$ is w^*-compact if and only if it is w^*-closed and norm bounded, thus the above formula makes sense for every $x^*, y^* \in K$.

● **PROBLEM 1.5.** Let $1 < p < \infty$. Then the unit ball of ℓ_p is metrisable in its weak topology. Justify this statement. Next, show that for any uncountable index set Γ the unit ball of $\ell_p(\Gamma)$ is not metrisable in its weak topology, although being weakly compact.

Remark. The last subordinate clause follows from the classical fact saying that a Banach space X is reflexive if and only if the unit ball B_X is weakly compact.

● **PROBLEM 1.6.** Let X be a Banach space. Show that in each of the two following cases X embeds isometrically into ℓ_∞:

(a) X is separable;
(b) X is a dual of a separable Banach space.

Generalise clauses (a) and (b) for arbitrarily large density characters of X and the predual of X, respectively.

Hint. Take what you get and write down what you want to have.

● **PROBLEM 1.7.** Show that for every non-empty index set Γ the space $\ell_\infty(\Gamma)$ is *isometrically injective*, that is, for any two Banach spaces $Y \subset X$ every bounded linear operator $t\colon Y \to \ell_\infty(\Gamma)$ admits an extension to a bounded linear operator $T\colon X \to \ell_\infty(\Gamma)$ with $\|t\| = \|T\|$. Next show that every isometrically injective Banach space is isometric to a 1-complemented subspace of $\ell_\infty(\Gamma)$, for some set Γ.

Hint. Apply the Hahn–Banach theorem. For the second assertion you should probably like to use Problem 1.6.

● **PROBLEM 1.8.** Prove the following characterisations of relative compactness in the spaces c_0 and ℓ_p $(1 \leqslant p < \infty)$:

(a) A set $K \subset c_0$ is relatively compact if and only if there exists a sequence $(x_n)_{n=1}^{\infty} \in c_0$ such that $|k_n| \leqslant |x_n|$ for every $n \in \mathbb{N}$ and $(k_n)_{n=1}^{\infty} \in K$.
(b) A bounded set $K \subset \ell_p$ is relatively compact if and only if

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} |k_j|^p = 0 \quad \text{uniformly for all } (k_n)_{n=1}^{\infty} \in K.$$

Hint. Use the fact that a subset of a complete metric space is relatively compact if and only if it is totally bounded.

● **PROBLEM 1.9.** Let X be a Banach space for which there is a surjective bounded linear operator $Q\colon X \to \ell_1$. Show that X contains a complemented subspace isomorphic to ℓ_1.

Hint. Use the Open Mapping Theorem. The crucial thing to show is that there exists a *lifting* for Q, that is, a bounded linear operator $R\colon \ell_1 \to X$ such that QR is the identity on ℓ_1.

Remark. The clause formulated in the hint means that ℓ_1 is *projective*. It is known that it is in fact the only one (up to isomorphism) separable, infinite-dimensional, projective Banach space. The proof involves the fact that for every separable Banach space Y there is a surjective bounded linear operator from ℓ_1 onto Y. Suppose we take Y to be any projective Banach space, potentially different from any isomorphic copy of ℓ_1. Then Y must contain ℓ_1 complemented. However, using a bit of the theory of Schauder bases one may show that every complemented infinite-dimensional subspace of ℓ_1 must itself be isomorphic to ℓ_1 (the corresponding statement is also true for c_0 and ℓ_p with $1 \leqslant p < \infty$; this was proved by A. Pełczyński in 1960).

More generally, the spaces $\ell_1(\Gamma)$, with Γ being any non-empty index set, exhaust the whole class of non-zero projective Banach spaces.

● **PROBLEM 1.10** (Helly's theorem). Given a Banach space X, functionals $x_1^*, \ldots, x_n^* \in X^*$ and scalars $\alpha_1, \ldots, \alpha_n$, show that the following assertions are equivalent:

(i) there exists $x \in X$ such that $x_j^* x = \alpha_j$ for $1 \leqslant j \leqslant n$;
(ii) there is a constant γ such that $\left|\sum_{j=1}^n \alpha_j \beta_j\right| \leqslant \gamma \left\|\sum_{j=1}^n \beta_j x_j^*\right\|$ for all scalars $\beta_1, \ldots, \beta_n$.

Show also that if condition (ii) holds true then for every $\varepsilon > 0$ the vector $x \in X$ in assertion (i) may be chosen so that $\|x\| \leqslant \gamma + \varepsilon$.

Conclude that for any $x^{**} \in X^{**}$, any finite-dimensional subspace E of X^* and any $\varepsilon > 0$ there exists $x \in X$ satisfying $\|x\| \leqslant \|x^{**}\| + \varepsilon$ and $x^{**}x^* = x^*x$ for every $x^* \in E$.

Hint. For the implication (ii)⇒(i) reduce the problem to the case where $x_1^*, \ldots, x_n^*$ are linearly independent. Next, use the following well-known algebraical fact: If $\Lambda_1, \ldots, \Lambda_n$ and Λ are linear functionals on a vector space, then Λ is a linear combination of Λ_j's if and only if $\bigcap_{j=1}^n \ker(\Lambda_j) \subseteq \ker(\Lambda)$ (see [W. Rudin, *Functional Analysis*, Lemma 3.9]); try to use the idea from the proof of this fact. This should be enough to derive clause (ii). In order to get also the estimate $\|x\| \leqslant \gamma + \varepsilon$, use the following simple corollary from the Hahn–Banach theorem: If Y is a closed subspace of a normed space Z and $x \in Z \setminus Y$, then there exists a continuous linear functional f on Z such that $\|f\| = 1$, $f(x) = \operatorname{dist}(x, Y)$ and $Y \subseteq \ker(f)$.

Remark. Observe that the last statement of this problem is tautologically true (with $E = X^*$ and $\varepsilon = 0$) in the case where X is reflexive. So, we may say that even non-reflexive spaces behave 'locally' reflexively. This is a special case of a deeper result called the *Principle of Local Reflexivity* which was proved by J. Lindenstrauss and H.P. Rosenthal in 1969.

● **PROBLEM 1.11.** Let X, Y be Banach spaces and $T\colon X \to Y$ be a bounded linear operator. Prove that the following assertions are equivalent:

(i) T maps weakly compact subsets to norm compact subsets;
(ii) T is weak-to-norm sequentially continuous, that is, for every weakly null sequence $(x_n)_{n=1}^\infty \subset X$ the sequence $(Tx_n)_{n=1}^\infty \subset Y$ converges to zero in norm;
(iii) T maps weakly Cauchy sequences to norm convergent sequences.

Hint. Concerning (i)⇒(ii): Always keep in mind that bounded linear operators are weak-to-weak continuous. Concerning (i)⇐(ii): Use the Eberlein–Šmulian theorem. For the implication (ii)⇒(iii) you need to develop a simple but delightful trick.

Remark. Recall that a sequence $(x_n)_{n=1}^\infty \subset X$ is called *weakly Cauchy* whenever for every $x^* \in X^*$ the limit $\lim_n x^*x_n$ exists (but may depend on x^*). Any operator $T\colon X \to Y$ satisfying one (hence, all) of the assertions (i)-(iii) is called *completely continuous* or a *Dunford–Pettis operator*.

The Eberlein–Šmulian theorem asserts that a subset A of a Banach space X is [relatively] weakly compact if and only if it is [relatively] weakly sequentially compact which happens if and only if it is [relatively] weakly countably compact. Relate it to the fact that the weak topology is never metrisable (unless the space in question is finite-dimensional), sometimes even on weakly compact subsets (see Problem 1.5), hence seemingly we are not permitted to verify weak compactness by sequential arguments. The above quoted theorem says that things are better than they seem. According to what Albiac and Kalton wrote, *The Eberlein–Šmulian theorem was probably the deepest result of earlier (pre-1950) Banach space theory.* Its proof may be found in almost every textbook on functional analysis.

● **PROBLEM 1.12.** Consider the set $A \subset \ell_2$ given by

$$A = \{e_m + me_n : m, n \in \mathbb{N},\, m < n\},$$

where as usual $(e_n)_{n=1}^{\infty}$ are the standard unit vectors. Show that 0 belongs to the weak closure of A although there is no weakly null sequence of elements from A.

Remark. This means that there is no analogue of the Eberlein–Šmulian theorem for weakly closed sets. The above example is due to J. von Neumann.

● **PROBLEM 1.13.** Prove that:

(a) for every $x \in c_0$ and every weakly null sequence $(y_n)_{n=1}^{\infty}$ in c_0 we have

$$\limsup_{n\to\infty} \|x + y_n\| = \max\Big\{\|x\|, \limsup_{n\to\infty} \|y_n\|\Big\};$$

(b) for every $x \in \ell_p$ (where $1 \leqslant p < \infty$) and every weakly null sequence $(y_n)_{n=1}^{\infty}$ in ℓ_p we have

$$\limsup_{n\to\infty} \|x + y_n\|^p = \|x\|^p + \limsup_{n\to\infty} \|y_n\|^p.$$

Hint. Try to first prove the assertion for some special sequences x.

Remark. These two tricky formulas have been used by S. Delpech to give a surprisingly short proof of Pitt's compactness theorem which asserts that for $1 \leqslant q < \infty$ every bounded linear operator from ℓ_p, $p > q$, or c_0 into ℓ_q is compact.

● **PROBLEM 1.14.** In each of the following cases prove or disprove that ℓ_∞ contains an isomorphic copy of X:

(a) $X = c_0(\mathfrak{c})$,
(b) $X = \ell_1(\mathfrak{c})$,
(c) $X = \ell_p(\mathfrak{c})$ for $1 < p < \infty$.

Hint. The following two ingredients would be useful: (i) ℓ_∞^* is w^*-separable being the bidual of the norm separable Banach space ℓ_1 (it follows immediately from Goldstine's theorem); (ii) if Y is a subspace of a Banach space Z, with Z^* being w^*-separable, then Y^* is also w^*-separable. Problem 1.6(b) would be useful to investigate the clause (b). You may also engage the reflexivity of $\ell_p(\mathfrak{c})$, for $1 < p < \infty$, and apply a couple of facts on adjoint operators.

Remark. Recall that the classical Goldstine theorem asserts that for any Banach space X the range $i(B_X)$ of the unit ball via the canonical embedding $i\colon X \to X^{**}$ is w^*-dense in $B_{X^{**}}$, the unit ball of X^{**}. From this it easily follows that $i(X)$, the range of the whole space, is w^*-dense in the whole of X^{**}. These two statements are very similar to the corrolary from Helly's theorem which is to be proved in Problem 1.10. In fact, once you solve this problem, with a very little additional effort you get Goldstine's theorem.

● **PROBLEM 1.15.** Prove 'elementarily' (that is, not appealing to the fact that ℓ_∞ is a von Neumann algebra) that the space ℓ_∞ has a unique predual, up to an isometric isomorphism.

Hint. For each $n \in \mathbb{N}$ let δ_n be the functional on ℓ_∞ which evaluates the nth coordinate. By using the Krein–Šmulian theorem, show that $\ker\delta_n$ is w^*-closed for every predual E of ℓ_∞ (i.e. it is closed in the topology $\sigma(X, E)$). So, E must contain $\overline{\mathrm{span}}\{\delta_n\}_{n=1}^\infty$. What is this space? Why E cannot be larger that this space?

Remark. Of course, in the above statement we mean that there is the unique predual ℓ_1 of ℓ_∞ in the class of Banach spaces because without requiring completeness the (dense) subspace of ℓ_1 consisting of all finitely supported sequences would also be such a predual.

Recall that the Krein–Šmulian theorem (in its particular case) asserts that a linear subspace Y of E^* is w^*-closed if and only if its intersection with the unit ball of E^*, the set $Y \cap B_{E^*}$, is w^*-compact.

● **PROBLEM 1.16.** Let $(x_n)_{n=1}^\infty$ be a weakly null sequence in a Banach space X and $(x_m^*)_{m=1}^\infty$ be a bounded sequence in X^*. Show that for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x_m^* x_{n_0}| < \varepsilon$ for infinitely many m's.

Hint. Try first to prove something different. Namely, that for every weakly null sequence $(y_n)_{n=1}^\infty$ and every $\delta > 0$ there exists a finite sequence $(\lambda_j)_{j=1}^k$ of positive numbers summing up to 1 such that

$$\max\left\{\left\|\sum_{j=1}^k \varepsilon_j \lambda_j y_j\right\| \colon |\varepsilon_j| = 1 \text{ for } 1 \leqslant j \leqslant k\right\} < \delta.$$